

Cesari-type Conditions for Semilinear Elliptic Equations with Leading Term Containing Controls *

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Abstract. An optimal control problem governed by semilinear elliptic partial differential equations is considered. The equation is in divergence form with the leading term containing controls. By studying the G -closure of the leading term, an existence result is established under a Cesari-type condition.

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1 Introduction

Consider the following controlled elliptic partial differential equation of divergence form:

$$\begin{cases} -\nabla \cdot (A(x, u(x)) \nabla y(x)) = f(x, y(x), u(x)), & \text{in } \Omega, \\ y(x) = 0, & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

where Ω is a smooth bounded domain in \mathbb{R}^n , $A : \Omega \times U \rightarrow \mathbb{R}^{n \times n}$ is a map taking values in the set of all positive definite matrices, and $f : \Omega \times \mathbb{R} \times U \rightarrow \mathbb{R}$, with U being a separable metric space. The control function $u(\cdot)$ is taken from the set

$$\mathcal{U} \equiv \{v : \Omega \rightarrow U \mid v(\cdot) \text{ is measurable}\}.$$

Let the cost functional be defined by

$$J(u(\cdot)) = \int_{\Omega} f^0(x, y(x), u(x)) dx, \quad (1.2)$$

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where $y(\cdot)$ is the solution of (1.1) (called the state corresponding to control $u(\cdot)$). Our optimal control problem is as follows.

Problem (C). Find a $\bar{u}(\cdot) \in \mathcal{U}$ such that

$$J(\bar{u}(\cdot)) = \inf_{u(\cdot) \in \mathcal{U}} J(u(\cdot)). \quad (1.3)$$

Any $\bar{u}(\cdot)$ satisfying (1.3) is called an optimal control. It is well-known that optimal control of Problem (C) may fail to exist. When $A(x, u) \equiv A(x)$, a suitable Cesari-type condition and some other mild conditions will guarantee the existence of an optimal control. Cesari-type condition is a natural generalization of optimal control problem with linear state equations and convex cost functionals. Many results are available along these lines. For further detail, see the books by Cesari [6], Li and Yong [11], for examples. For the two phrase case, i.e., $U = \{0, 1\}$ and $A(x, i) \equiv A_i$ ($i = 0, 1$) with A_0, A_1 being two constant matrices, Murat and Tartar gave an existence result in the framework of “relaxation” control (see [16]). However, it seems no work devoted to the existence of optimal controls for general cases.

In this paper, we will give a Cesari-type result to ensure the existence of a solution to Problem (C). We always assume Λ and λ be two constants satisfying $\Lambda \geq \lambda > 0$. Denote by \mathcal{S}_+^n the set of all $n \times n$ (symmetric) positive definite matrices and

$$\mathcal{M}_{\Lambda, \lambda} = \left\{ Q \in \mathcal{S}_+^n \mid \lambda |\xi|^2 \leq Q\xi \cdot \xi \leq \Lambda |\xi|^2, \quad \forall \xi \in \mathbb{R}^n \right\}.$$

For a matrix B , we always denote B_{ij} as its entries.

We recall that a Polish space is a separable completely metrizable topological space. We mention that all (nonempty) closed sets and open sets in \mathbb{R}^m are polish spaces.

We make the following assumptions.

(S1) Set Ω is a bounded domain in \mathbb{R}^n with a C^2 boundary $\partial\Omega$.

(S2) U is a Polish space.

(S3) Function $A(x, v)$ takes values in $\mathcal{M}_{\Lambda, \lambda}$, which are measurable in $x \in \Omega$ and continuous in $v \in U$. Further, there exists an $F \in L^\infty(\Omega; \mathbb{R}^m)$ and a continuous $\omega : [0, +\infty) \rightarrow [0, +\infty)$, such that $\omega(0) = 0$ and

$$|A(x, v) - A(\tilde{x}, v)| \leq \omega(|F(x) - F(\tilde{x})|), \quad \forall x, \tilde{x} \in \Omega, v \in U. \quad (1.4)$$

(S4) Function $f(x, y, v)$ is measurable in x and continuous in $(y, v) \in \mathbb{R} \times U$ for almost all $x \in \Omega$. Moreover, for almost all $x \in \Omega$,

$$f_y(x, y, v) \leq 0, \quad \forall (y, v) \in \mathbb{R} \times U, \quad (1.5)$$

and for any $R > 0$, there exists an $M_R > 0$ such that

$$|f(x, y, v)| + |f_y(x, y, v)| \leq M_R, \quad \forall v \in U, |y| \leq R. \quad (1.6)$$

(S5) Function $f^0(x, y, v)$ is measurable in x , lower semicontinuous in $(y, v) \in \mathbb{R} \times U$ for almost all $x \in \Omega$. Moreover, for almost all $x \in \Omega$ and for any $R > 0$, there exists an $K_R > 0$ such that

$$f^0(x, y, v) \geq -K_R, \quad \forall v \in U, |y| \leq R. \quad (1.7)$$

Remark 1.1. In (1.4), m need not equal to n . We can see that (1.4) holds naturally when U is a finite set. On the other hand, if $A(x, u)$ is uniformly continuous in $x \in \Omega$ with respect to $u \in U$, then (1.4) holds.

Remark 1.2. Without loss of generality, we can suppose that $\omega(\cdot)$ is a continuous module in (S3), i.e.,

- (i) $\omega(\cdot)$ is continuous and increasing on $[0, +\infty)$,
- (ii) $\omega(0) = 0$,
- (iii) it holds that

$$\omega(r + s) \leq \omega(r) + \omega(s), \quad \forall r, s > 0.$$

In fact, if necessary, we can replacing $\omega(\cdot)$ by

$$\tilde{\omega}(r) \triangleq \sup_{\substack{|s-t| \leq r \\ s, t \in [0, R]}} |\omega(s) - \omega(t)|, \quad r \geq 0,$$

where $R = 2\|F\|_{L^\infty(\Omega; \mathbb{R}^m)}$.

Denote $Z = [0, 1]^n$ and

$$\mathcal{U}_Z \equiv \{v : Z \rightarrow U \mid v(\cdot) \text{ is measurable}\}.$$

Let e_1, e_2, \dots, e_n be the canonical basis of \mathbb{R}^n . We call a function $g(x)$ is Z -periodic if it admits periodic 1 in the direction e_j ($j = 1, 2, \dots, n$). Denote

$$\begin{aligned} L_\#^\infty(Z) &= \{h \in L^\infty(\mathbb{R}^n) \mid h \text{ is } Z\text{-periodic}\}, \\ H_\#^1(Z) &= \{h \in H_{loc}^1(\mathbb{R}^n) \mid h \text{ is } Z\text{-periodic}\}. \end{aligned}$$

We define

$$\mathcal{E}(x, y) = \left\{ (P, \zeta, \zeta^0) \in \mathcal{S}_+^n \times \mathbb{R} \times \mathbb{R} \mid P = A(x, u), \zeta = f(x, y, u), \zeta^0 \geq f^0(x, y, u), u \in U \right\} \quad (1.8)$$

and

$$\begin{aligned} G\mathcal{E}(x, y) &= \left\{ (P, \zeta, \zeta^0) \in \mathcal{S}_+^n \times \mathbb{R} \times \mathbb{R} \mid P_{ij} = \int_Z A(x, u(z))(e_i + \nabla w^i(z; x)) \cdot e_j \, dz, \right. \\ &\quad \text{where } w^i(\cdot) \in H_{\#}^1(Z) \text{ solves } \nabla_z \cdot \left(A(x, u(z))(e_i + \nabla w^i(z; x)) \right) = 0, \\ &\quad \left. \zeta = \int_Z f(x, y, u(z)) \, dz, \zeta^0 \geq \int_Z f^0(x, y, u(z)) \, dz, u(\cdot) \in \mathcal{U}_Z \right\}. \end{aligned} \quad (1.9)$$

Our main result is the following theorem.

Theorem 1.1. *Assume (S1) — (S5), and the following condition hold*

$$\mathcal{E}(x, y) = \bigcap_{\delta > 0} \overline{G\mathcal{E}(x, B_\delta(y))}, \quad \text{a.e. } (x, y) \in \Omega \times \mathbb{R}, \quad (1.10)$$

where $\mathcal{E}(x, y)$ and $G\mathcal{E}(x, y)$ are defined by (1.8) and (1.9), $\overline{G\mathcal{E}(x, y)}$ is the closure of $G\mathcal{E}(x, y)$ in $\mathcal{S}_+^n \times \mathbb{R} \times \mathbb{R}$, and $B_\delta(y)$ is the ball centered at y with radius δ . Then Problem (C) admits at least one solution.

Remark 1.3. *When the leading term is independent of control variable, i.e. $A(x, u) \equiv A(x)$, Theorem 1.1 is equivalent to the classical existence result of optimal control (see Theorem 6.4 in Chapter 3 of [11]). This fact will follow by Proposition 3.4 in Section 3.*

When dealing with problems with controls containing in the leading term, we meet a main difficulty that is to find the state equation corresponding to the weak limit of state sequence. This is involved with the H -convergence and G -closure problem. It is known that optimal control usually does not exist for Problem (C) and therefore to seek optimal relaxed control for Problem (C) is more meaningful than to seek a solution for Problem (C). Nevertheless, we think this paper contains some useful ideas for us to get the relaxation of Problem (C), which will be our forthcoming work. In this paper, we will give a local representation of G -closure in Section 2, which is critical in proving the existence theorem. While Section 3 is devoted to a proof of Theorem 1.1 and some propositions.

2 H-convergence and Local Representation of G-closure

Now, let us recall the notion of H -convergence. This kind of convergence was introduced by Murat and Tartar in [15].

Definition 2.1. *A sequence of matrix valued functions $A^\varepsilon(\cdot) \in L^\infty(\Omega; \mathcal{M}_{\Lambda, \lambda})$ is said to H -converge to a matrix valued function $A^*(\cdot) \in L(\Omega; \mathcal{M}_{\Lambda, \lambda})$, if for any right hand side $f \in H^{-1}(\Omega)$,*

the sequence $y^\varepsilon(\cdot) \in H_0^1(\Omega)$ of weak solutions of

$$\begin{cases} -\nabla \cdot (A^\varepsilon(x) \nabla y^\varepsilon(x)) = f, & \text{in } \Omega, \\ y^\varepsilon(x) = 0, & \text{on } \partial\Omega \end{cases} \quad (2.1)$$

satisfies

$$y^\varepsilon(\cdot) \rightharpoonup \bar{y}(\cdot), \quad \text{weakly in } H_0^1(\Omega),$$

where $\bar{y}(\cdot)$ is the weak solution of

$$\begin{cases} -\nabla \cdot (A^*(x) \nabla \bar{y}(x)) = f, & \text{in } \Omega, \\ \bar{y}(x) = 0, & \text{on } \partial\Omega. \end{cases} \quad (2.2)$$

The notion of “H-convergence” makes sense because of the next compactness proposition (see Theorem 2 in [15]).

Proposition 2.2. *For any sequence $A^\varepsilon(\cdot)$ of matrices in $L^\infty(\Omega; \mathcal{M}_{\Lambda, \lambda})$, there exists a subsequence of $A^\varepsilon(\cdot)$, H -converges to an $A^*(\cdot) \in L^\infty(\Omega; \mathcal{M}_{\Lambda, \lambda})$.*

This proposition proves the existence of an H -limit for a subsequence of a bounded sequence, but it delivers no explicit formula for this limit. The next proposition shows that when $A^\varepsilon(\cdot) = A(\frac{\cdot}{\varepsilon})$ with some periodic matrix valued function $A(\cdot)$, $A^\varepsilon(\cdot)$ H -converges to an H -limit defined by an explicit formula (up to solving some corresponding cell problems). The proposition can be stated as

Proposition 2.3. *Let $A(\cdot) \in L^\infty_\#(Z; \mathcal{M}_{\Lambda, \lambda})$. Then*

$$A\left(\frac{\cdot}{\varepsilon}\right) \xrightarrow{H} A^* \chi_\Omega(\cdot)$$

with $A^* \in \mathcal{M}_{\Lambda, \lambda}$ being a constant matrix defined by its entries

$$A_{ij}^* = \int_Z A(z)(e_i + \nabla w_i) \cdot e_j \, dz, \quad (2.3)$$

where $\{w_i\}_{1 \leq i \leq n}$ is the family of unique solutions in $H_\#^1(Z)/\mathbb{R}$ of the cell problems

$$-\nabla \cdot (A(z)(e_i + \nabla w_i(z))) = 0, \quad \text{in } Z. \quad (2.4)$$

For a proof of the above proposition, see Theorem 1.3.18 of [2] or Theorem 1.3.1 of [3]. The next classical result (see Theorem 1.3.23 in [2], for example) shows the fact that a general H -limit $A^*(\cdot)$ can be attained as the limit of a sequence of periodic homogenized matrices.

Proposition 2.4. Assume $A^\varepsilon(\cdot) \in L^\infty(\Omega; \mathcal{M}_{\Lambda, \lambda})$ H -converge to a limit $A^*(\cdot)$. For any x in Ω and any sufficiently small positive $h > 0$, let $A_{\varepsilon, h}^*(\cdot)$ be the periodic homogenized matrix defined by its entries

$$(A_{\varepsilon, h}^*(x))_{ij} = \int_Z A^\varepsilon(x + hz)(e_i + \nabla w_{\varepsilon, h}^i(z; x)) \cdot e_j dz, \quad (2.5)$$

where $\{w_{\varepsilon, h}^i(\cdot; x)\}_{1 \leq i \leq n}$ is the family of unique solutions in $H_\#^1(Z)/\mathbb{R}$ of the cell problems

$$-\nabla_z \cdot (A^\varepsilon(x + hz)(e_i + \nabla_z w_{\varepsilon, h}^i(z; x))) = 0, \quad \text{in } Z. \quad (2.6)$$

Then, along a subsequence $h \rightarrow 0$,

$$\lim_{\varepsilon \rightarrow 0^+} A_{\varepsilon, h}^*(x) \rightarrow A^*(x), \quad \text{a.e. } x \in \Omega.$$

We list some useful properties of H -convergence in follows. For proofs of these results, see Proposition 1.2.18, Proposition 1.2.22 and Proposition 1.3.44 in [2].

Proposition 2.5. Let $A^\varepsilon(\cdot)$ and $B^\varepsilon(\cdot)$ be two sequences in $L^\infty(\Omega; \mathcal{M}_{\Lambda, \lambda})$, which H -converge to $A^*(\cdot)$ and $B^*(\cdot)$, respectively. Let Ω_0 be an open subset of Ω . If

$$A^\varepsilon(x) = B^\varepsilon(x), \quad \text{in } \Omega_0,$$

then

$$A^*(x) = B^*(x), \quad \text{in } \Omega_0.$$

Proposition 2.5 shows that the value of H -limit $A^*(\cdot)$ in a region Ω_0 does not depend on the values of sequence $A^\varepsilon(\cdot)$ outside of this region, which is precisely what we mean by locality.

Proposition 2.6. Assume $A^\varepsilon(\cdot) \in L^\infty(\Omega; \mathcal{M}_{\Lambda, \lambda})$ converge strongly to a limit matrix $A^*(\cdot) \in L^1(\Omega; \mathcal{M}_{\Lambda, \lambda})$. Then, $A^\varepsilon(\cdot)$ H -converges to $A^*(\cdot)$ too.

In particular, if $A^\varepsilon(\cdot) \in \mathcal{M}_{\Lambda, \lambda}$ converges to $A^*(\cdot)$ almost everywhere in Ω , then $A^\varepsilon(\cdot)$ H -converges to $A^*(\cdot)$.

This proposition shows that H -convergence is weaker than strong convergence. On the other hand, it is well-known that usually the weak limit of a sequence $A^\varepsilon(\cdot)$ does not equal to its H -limit.

Proposition 2.7. Let (S1) hold. Then there exist constants $C > 0$ and $\delta > 0$ such that, for any $1 \leq p \leq 1 + \delta$ and two sequences of $A^\varepsilon(\cdot)$ and $B^\varepsilon(\cdot)$ in $L^\infty(\Omega; \mathcal{M}_{\Lambda, \lambda})$, which H -converge to $A^*(\cdot)$ and $B^*(\cdot)$, respectively, it holds that

$$\|A^*(\cdot) - B^*(\cdot)\|_{L^p(\Omega)} \leq C \lim_{\varepsilon \rightarrow 0^+} \|A^\varepsilon(\cdot) - B^\varepsilon(\cdot)\|_{L^p(\Omega)}. \quad (2.7)$$

Now define

$$\mathcal{G}(A) = \left\{ P(\cdot) \in L^\infty(\Omega; \mathcal{M}_{\Lambda, \lambda}) \mid \exists u^\varepsilon(\cdot) \in \mathcal{U}, \text{ s.t. } A(\cdot, u^\varepsilon(\cdot)) \xrightarrow{H} A^*(\cdot) \right\}. \quad (2.8)$$

We see that $\mathcal{G}(A)$ is the set of all possible H -limits of $\{A(\cdot, u(\cdot))\}_{u(\cdot) \in \mathcal{U}}$. A very important problem called G -closure problem is to find out the structure of $\mathcal{G}(A)$. Many works devoted to this problem dealt with two-phase composite cases (see, for examples, [4], [14] and [17]). In [17], a precise formula of $\mathcal{G}(A)$ was given for a special two-phase case of A taking only αI and βI for some $\beta > \alpha > 0$. Unfortunately, in most cases including usual two-phase cases, precise knowledge of the G -closure are still lacking.

A local representation of $\mathcal{G}(A)$ is crucial to our main result. We give a simple lemma related to Assumption (S3) first.

Lemma 2.8. *Let $\omega(\cdot)$ be continuous on $[0, +\infty)$, $\omega(0) = 0$ and $F(\cdot) \in L^\infty(\Omega; \mathbb{R}^m)$. We have the following results.*

(i)

$$\lim_{h \rightarrow 0} \int_Z \omega(|F(x + hz) - F(x)|) dz = 0, \quad \text{a.e. } x \in \Omega. \quad (2.9)$$

(ii) Let $\{\Omega_j^k\}_{1 \leq j \leq k}$ be a family of measurable decompositions of Ω such that:

- (a) if $i \neq j$, $\Omega_i^k \cap \Omega_j^k = \emptyset$;
- (b) for any k , $\bigcup_{1 \leq j \leq k} \Omega_j^k = \Omega$;
- (c) $\lim_{k \rightarrow +\infty} \max_{1 \leq j \leq k} \text{diam}(\Omega_j^k) = 0$.

Then

$$\lim_{k \rightarrow \infty} \sum_{\ell=1}^k \frac{1}{|\Omega_\ell^k|} \int_{\Omega_\ell^k} \int_{\Omega_\ell^k} \omega(|F(s) - F(x)|) ds dx = 0, \quad (2.10)$$

where $|E|$, $\text{diam}(E)$ denotes the Lebesgue measure and the diameter of E , respectively.

Proof. (i) Let $x \in \Omega$ be a Lebesgue point of $F(\cdot)$ and satisfy $|F(x)| \leq \|F\|_{L^\infty(\Omega; \mathbb{R}^m)}$. Then

$$\lim_{h \rightarrow 0} \int_Z |F(x + hz) - F(x)| dz = 0.$$

Thus, as a function of z , $|F(x + hz) - F(x)|$ converges in measure to 0 as $h \rightarrow 0$. Since

$$\omega(|F(x + hz) - F(x)|) \leq \max \{ \omega(r) \mid 0 \leq r \leq 2\|F\|_{L^\infty(\Omega; \mathbb{R}^m)} \}, \quad \text{a.e. } z \in Z,$$

we get (2.9) by Lebesgue's dominated convergence theorem.

(ii) By Remark 1.2, we suppose $\omega(\cdot)$ is a continuous module without loss of generality. For any $\Phi(\cdot) \in C(\overline{\Omega}; \mathbb{R}^m)$, we have

$$\begin{aligned}
& \sum_{\ell=1}^k \frac{1}{|\Omega_\ell^k|} \int_{\Omega_\ell^k} \int_{\Omega_\ell^k} \omega(|F(s) - F(x)|) ds dx \\
& \leq \sum_{\ell=1}^k \frac{1}{|\Omega_\ell^k|} \int_{\Omega_\ell^k} \int_{\Omega_\ell^k} \left\{ \omega(|F(s) - \Phi(s)|) + \omega(|\Phi(s) - \Phi(x)|) + \omega(|\Phi(x) - F(x)|) \right\} ds dx \\
& = 2 \int_{\Omega} \omega(|F(x) - \Phi(x)|) dx + \sum_{\ell=1}^k \frac{1}{|\Omega_\ell^k|} \int_{\Omega_\ell^k} \int_{\Omega_\ell^k} \omega(|\Phi(s) - \Phi(x)|) ds dx.
\end{aligned}$$

Consequently, if we set $F(x) = 0$ for $x \notin \Omega$ and choose

$$\Phi(x) = \int_Z F(x + hz) dz,$$

we have $\Phi \in C(\overline{\Omega}; \mathbb{R}^m)$. Consequently, it follows easily from the uniform continuity of Φ and the assumption (c) that

$$\lim_{k \rightarrow +\infty} \sum_{\ell=1}^k \frac{1}{|\Omega_\ell^k|} \int_{\Omega_\ell^k} \int_{\Omega_\ell^k} \omega(|\Phi(s) - \Phi(x)|) ds dx = 0.$$

Thus,

$$\begin{aligned}
& \overline{\lim}_{k \rightarrow +\infty} \sum_{\ell=1}^k \frac{1}{|\Omega_\ell^k|} \int_{\Omega_\ell^k} \int_{\Omega_\ell^k} \omega(|F(s) - F(x)|) ds dx \\
& \leq 2 \int_{\Omega} \omega\left(|F(x) - \int_Z F(x + hz) dz|\right) dx \\
& \leq 2 \int_{\Omega} \omega\left(\int_Z |F(x + hz) - F(x)| dz\right) dx, \quad \forall h > 0.
\end{aligned}$$

Therefore,

$$\begin{aligned}
& \overline{\lim}_{k \rightarrow +\infty} \sum_{\ell=1}^k \frac{1}{|\Omega_\ell^k|} \int_{\Omega_\ell^k} \int_{\Omega_\ell^k} \omega(|F(s) - F(x)|) ds dx \\
& \leq 2 \lim_{h \rightarrow 0} \int_{\Omega} \omega\left(\int_Z |F(x + hz) - F(x)| dz\right) dx = 0.
\end{aligned}$$

We get the proof. \square

Now, we will give a local representation of $\mathcal{G}(A)$.

Theorem 2.9. *Assume (S1)–(S3) hold. Then the G -closure set $\mathcal{G}(A)$ is characterized by*

$$\mathcal{G}(A) = \left\{ P(\cdot) \in L^\infty(\Omega; \mathcal{M}_{\Lambda, \lambda}) \mid P(x) \in \overline{G_x(A)}, \text{ a.e. } x \in \Omega \right\} \quad (2.11)$$

with $G_x(A)$ being defined by

$$G_x(A) = \left\{ Q \in \mathcal{S}_+^n \mid \exists u \in \mathcal{U}_Z, \text{ s.t. } Q_{ij} = \int_Z A(x, u(z))(e_i + \nabla w^i(z; x)) \cdot e_j dz, \right. \\ \left. \text{where } w^i(\cdot; x) \in H_{\#}^1(Z), \nabla_z \cdot \left(A(x, u(z))(e_i + \nabla w^i(z; x)) \right) = 0 \right\}. \quad (2.12)$$

Remark 2.1. It is easy to see that in (2.12), $G_x(A)$ can be rewritten as

$$G_x(A) = \left\{ Q \in \mathcal{S}_+^n \mid \exists u \in \mathcal{U}_Z, \text{ s.t. } Q_{ij} = \int_Z A(x, u(z))(e_i + \nabla w^i(z; x)) \cdot (e_j + \nabla w^j(z)) dz, \right. \\ \left. \text{where } w^i(\cdot; x) \in H_{\#}^1(Z), \nabla_z \cdot \left(A(x, u(z))(e_i + \nabla w^i(z; x)) \right) = 0 \right\}, \quad (2.13)$$

which implies $G_x(A) \subseteq \mathcal{M}_{\Lambda, \lambda}$.

Proof of Theorem 2.9. Denote

$$\mathcal{P}(A) = \left\{ P(\cdot) \in L^\infty(\Omega; \mathcal{M}_{\Lambda, \lambda}) \mid P(x) \in \overline{G_x(A)}, \text{ a.e. } x \in \Omega \right\}.$$

We need to show $\mathcal{G}(A) = \mathcal{P}(A)$.

We prove $\mathcal{G}(A) \subseteq \mathcal{P}(A)$ first. Assume $A^*(\cdot) \in \mathcal{G}(A)$. Then there exists a sequence $u^\varepsilon(\cdot) \in \mathcal{U}$, such that as $\varepsilon \rightarrow 0^+$,

$$A(\cdot, u^\varepsilon(\cdot)) \xrightarrow{H} A^*(\cdot).$$

By Proposition 2.4, along a subsequence $h \rightarrow 0$,

$$A_{ij}^*(x) = \lim_{h \rightarrow 0} \lim_{\varepsilon \rightarrow 0} (\tilde{A}_{h, \varepsilon}^*(x))_{ij} \quad \text{a.e. } x \in \Omega, \quad (2.14)$$

where $\tilde{A}_{h, \varepsilon}^*(\cdot)$ is defined by

$$(\tilde{A}_{h, \varepsilon}^*(x))_{ij} = \int_Z A(x + hz, u^\varepsilon(x + hz))(e_i + \nabla_z \tilde{w}_{h, \varepsilon}^i(z; x)) \cdot e_j dz \quad (2.15)$$

with $\tilde{w}_{h, \varepsilon}^i(\cdot; x) \in H_{\#}^1(Z)/\mathbb{R}$ being the unique Z -periodic solution of

$$\nabla_z \cdot \left(A(x + hz, u^\varepsilon(x + hz))(e_i + \nabla_z \tilde{w}_{h, \varepsilon}^i(z; x)) \right) = 0. \quad (2.16)$$

On the other hand, define $A_{h, \varepsilon}^*(\cdot)$ by

$$(A_{h, \varepsilon}^*(x))_{ij} = \int_Z A(x, u^\varepsilon(x + hz))(e_i + \nabla_z w_{h, \varepsilon}^i(z; x)) \cdot e_j dz \quad (2.17)$$

with $w_{h, \varepsilon}^i(\cdot; x) \in H_{\#}^1(Z)/\mathbb{R}$ being the unique Z -periodic solution of

$$\nabla_z \cdot \left(A(x, u^\varepsilon(x + hz))(e_i + \nabla_z w_{h, \varepsilon}^i(z; x)) \right) = 0. \quad (2.18)$$

Then, combining (2.16) with (2.18), we get

$$\begin{aligned} & \nabla_z \cdot \left(A(x, u^\varepsilon(x + hz)) (\nabla \tilde{w}_{h,\varepsilon}^i(z; x) - \nabla w_{h,\varepsilon}^i(z; x)) \right) \\ = & \nabla_z \cdot \left[\left(A(x, u^\varepsilon(x + hz)) - A(x + hz, u^\varepsilon(x + hz)) \right) (e_i + \nabla_z \tilde{w}_{h,\varepsilon}^i(z; x)) \right]. \end{aligned} \quad (2.19)$$

Multiplying (2.19) by $\tilde{w}_{h,\varepsilon}^i(z; x) - w_{h,\varepsilon}^i(z; x)$ and using integration by part, we get from the periodicities of $\tilde{w}_{h,\varepsilon}^i(\cdot; x)$ and $w_{h,\varepsilon}^i(\cdot; x)$ that

$$\begin{aligned} & \int_Z A(x, u^\varepsilon(x + hz)) (\nabla \tilde{w}_{h,\varepsilon}^i(z; x) - \nabla w_{h,\varepsilon}^i(z; x)) \cdot (\nabla \tilde{w}_{h,\varepsilon}^i(z; x) - \nabla w_{h,\varepsilon}^i(z; x)) dz \\ = & \int_Z \left(A(x, u^\varepsilon(x + hz)) - A(x + hz, u^\varepsilon(x + hz)) \right) \\ & (e_i + \nabla_z \tilde{w}_{h,\varepsilon}^i(z; x)) \cdot (\nabla_z \tilde{w}_{h,\varepsilon}^i(z; x) - \nabla_z w_{h,\varepsilon}^i(z; x)) dz. \end{aligned} \quad (2.20)$$

Then the ellipticity of A yields

$$\begin{aligned} & \lambda \|\nabla w_{h,\varepsilon}^i(\cdot; x) - \nabla \tilde{w}_{h,\varepsilon}^i(\cdot; x)\|_{L^2(Z)} \\ \leq & \left\{ \int_Z \left| \left(A(x, u^\varepsilon(x + hz)) - A(x + hz, u^\varepsilon(x + hz)) \right) (e_i + \nabla w_{h,\varepsilon}^i(z; x)) \right|^2 dz \right\}^{1/2}. \end{aligned}$$

By (2.18) and Meyers' theorem (see [13], see also Theorem 1.3.41 and Remark 1.3.42 in [2]), there exist constants $p > 2$ and $C > 0$, both dependent only on λ , Λ and Ω , such that

$$\|\nabla \tilde{w}_{h,\varepsilon}^i(\cdot; x)\|_{L^p(Z)} \leq C, \quad \|\nabla w_{h,\varepsilon}^i(\cdot; x)\|_{L^p(Z)} \leq C. \quad (2.21)$$

Thus

$$\|\nabla w_{h,\varepsilon}^i(\cdot; x) - \nabla \tilde{w}_{h,\varepsilon}^i(\cdot; x)\|_{L^2(Z)} \leq C \left(\int_Z |A(x, u^\varepsilon(x + hy)) - A(x + hy, u^\varepsilon(x + hy))|^q dy \right)^{1/q}, \quad (2.22)$$

where $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$. Then it follows from (2.22) and (S3) that

$$\begin{aligned} & |(\tilde{A}_{h,\varepsilon}^*(x))_{ij} - (A_{h,\varepsilon}^*(x))_{ij}| \\ \leq & \int_Z |A(x, u^\varepsilon(x + hz)) (\nabla_z \tilde{w}_{h,\varepsilon}^i(z; x) - \nabla_z w_{h,\varepsilon}^i(z; x))| dz \\ & + \int_Z \left| \left(A(x + hz, u^\varepsilon(x + hz)) - A(x, u^\varepsilon(x + hz)) \right) (e_i + \nabla_z \tilde{w}_{h,\varepsilon}^i(z; x)) \right| dz \\ \leq & C \|\nabla \tilde{w}_{h,\varepsilon}^i(\cdot; x) - \nabla w_{h,\varepsilon}^i(\cdot; x)\|_{L^2(Z)} \\ & + C \left(\int_Z |A(x + hz, u^\varepsilon(x + hz)) - A(x, u^\varepsilon(x + hz))|^q dz \right)^{1/q} \|e_i + \nabla \tilde{w}_{h,\varepsilon}^i(\cdot; x)\|_{L^p(Z)} \\ \leq & C \left(\int_Z |A(x + hz, u^\varepsilon(x + hz)) - A(x, u^\varepsilon(x + hz))|^q dz \right)^{1/q} \\ \leq & C \left\{ \int_Z \left[\omega(|F(x + hz) - F(x)|) \right]^q dz \right\}^{1/q}. \end{aligned} \quad (2.23)$$

Thus, by Lemma 2.8, we get from (2.14) and (2.23) that along a subsequence $h \rightarrow 0$,

$$A_{ij}^*(x) = \lim_{h \rightarrow 0} \lim_{\varepsilon \rightarrow 0} (A_{h,\varepsilon}^*(x))_{ij}, \quad \text{a.e. } x \in \Omega. \quad (2.24)$$

Noting that $A_{h,\varepsilon}^*(x) \in G_x(A)$, we get $A^*(x) \in \overline{G_x(A)}$, a.e. $x \in \Omega$. Therefore $\mathcal{G}(A) \subseteq \mathcal{P}(A)$.

Next, we turn to prove $\mathcal{P}(A) \subseteq \mathcal{G}(A)$. Let $\{\Omega_j^k\}_{1 \leq j \leq k}$ be a family of measurable decompositions of Ω satisfying (a)–(c) in Lemma 2.8. Denote by $\chi_j^k(\cdot)$ the characteristic function of Ω_j^k .

We will show the result in three steps.

Step I. Assume $A(x, u) \equiv A(u)$.

Denote $G(A) \equiv G_x(A)$ since $G_x(A)$ is independent of x in this case.

For any $\tilde{A} \in G(A)$, we have $u(\cdot) \in \mathcal{U}_Z$ such that

$$\tilde{A}_{ij} = \int_Z A(u(z))(e_i + \nabla w^i(z)) \cdot e_j \, dz,$$

where $w^i(\cdot) \in H_{\#}^1(Z)/\mathbb{R}$ solves

$$\nabla \cdot (A(u(z))(e_i + \nabla w^i(z))) = 0.$$

By Proposition 2.3, $A(u(\frac{\cdot}{\varepsilon})) \xrightarrow{H} \tilde{A}\chi_{\Omega}(\cdot)$ as $\varepsilon \rightarrow 0^+$. Thus, $\tilde{A}\chi_{\Omega}(\cdot) \in \mathcal{G}(A)$. We denote this result simply by $G(A) \subseteq \mathcal{G}(A)$. Obviously, we can get $\overline{G(A)} \subseteq \mathcal{G}(A)$ immediately.

Let $A^*(\cdot) \in \mathcal{P}(A)$. Then

$$A^*(x) \in \overline{G(A)}, \quad \text{a.e. } x \in \Omega.$$

Define

$$\hat{A}_j^k = \frac{1}{|\omega_j^k|} \int_{\omega_j^k} A^*(x) \, dx, \quad \hat{A}_k(\cdot) = \sum_{j=1}^k \hat{A}_j^k \chi_j^k(\cdot).$$

Then

$$\hat{A}^k(\cdot) \rightarrow A^*(\cdot) \quad \text{strongly in } L^p(\Omega), \quad \forall 1 \leq p < \infty. \quad (2.25)$$

Denote

$$\mathcal{A}_j^k \equiv \left\{ Q \in \overline{G(A)} \mid |Q - \hat{A}_j^k| = \inf_{P \in \overline{G(A)}} |P - \hat{A}_j^k| \right\}, \quad 1 \leq j \leq k; \, k = 1, 2, 3, \dots \quad (2.26)$$

Since $\overline{G(A)}$ is closed, \mathcal{A}_j^k is always nonempty. Thus, we can select a constant matrix A_j^k from \mathcal{A}_j^k . Define

$$A_k(\cdot) = \sum_{j=1}^n A_j^k \chi_j^k(\cdot). \quad (2.27)$$

Since for almost all $x \in \Omega_j^k$, $A^*(x) \in \overline{G(A)}$, by (2.26), there is

$$\begin{aligned} & |A_k(x) - A^*(x)| \\ & \leq |A_j^k - \widehat{A}_j^k| + |\widehat{A}_j^k - A^*(x)| \\ & \leq 2|\widehat{A}_j^k - A^*(x)|. \end{aligned}$$

Thus, by (2.25),

$$A_k(\cdot) \rightarrow A^*(\cdot), \quad \text{strongly in } L^1(\Omega). \quad (2.28)$$

Consequently, by Proposition 2.6, we have

$$A_k(\cdot) \xrightarrow{H} A^*(\cdot). \quad (2.29)$$

The advantage of replacing \widehat{A}_k by A_k is that we have $A_j^k \in \overline{G(A)} \subseteq \mathcal{G}(A)$ while we do not always have $\widehat{A}_j^k \in \overline{G(A)}$.

Then, by Proposition 2.5 (local property), $A_k(\cdot) \in \mathcal{G}(A)$. Finally, by (2.29), $A^*(\cdot) \in \mathcal{G}(A)$. That is, $\mathcal{P}(A) \subseteq \mathcal{G}(A)$.

Step II. Assume $A(x, u) = \sum_{1 \leq j \leq k} A_j(u) \chi_j^k(x)$.

By what we have proved in Step I and the local property of H -convergence, we can see that $\mathcal{P}(A) \subseteq \mathcal{G}(A)$ holds in this case.

Step III. General cases. Let $A^*(\cdot) \in \mathcal{P}(A)$. Then $A^*(x) \in \overline{G_x(A)}$, a.e. $x \in \Omega$. We want to prove $A^*(x) \in \mathcal{G}(A)$. Without loss of generality, we can suppose that

$$A^*(x) \in G_x(A), \quad \forall x \in \Omega. \quad (2.30)$$

Define

$$\begin{cases} A_k(x, u) = \sum_{j=1}^k A_j^k(u) \chi_j^k(x), \\ A_j^k(u) = \frac{1}{|\Omega_j^k|} \int_{\Omega_j^k} A(s, u) ds, \end{cases} \quad (x, u) \in \Omega \times U.$$

While $A_k^*(\cdot)$ is a measurable selection of the projection of $A^*(x)$ on $\overline{G_x(A_k)}$, i.e., $A_k^*(\cdot)$ is measurable and

$$|A_k^*(x) - A^*(x)| = \inf_{P \in \overline{G_x(A_k)}} |P - A^*(x)|, \quad A_k^*(x) \in \overline{G_x(A_k)},$$

where $G_x(A_k)$ is defined by (2.12). By Filippov's lemma (see [10], or Corollary 2.26 of Chapter 3 in [11]), such an $A_k^*(\cdot)$ exists.

Now we will show that

$$A_k^*(\cdot) \rightarrow A^*(\cdot), \quad \text{strongly in } L^1(\Omega; \mathcal{S}_+^n). \quad (2.31)$$

By (2.30), there exists a $u(\cdot) \in \mathcal{U}_Z$, such that

$$(A^*(x))_{ij} = \int_Z A(x, u(z))(e_i + \nabla_z w^i(z; x)) \cdot e_j \, dz,$$

where for any $x \in \Omega$, $w^i(\cdot; x) \in H_{\#}^1(Z)/\mathbb{R}$ is the solution of

$$\nabla_z \cdot \left(A(x, u(z))(e_i + \nabla_z w^i(z; x)) \right) = 0.$$

Next, we can define $\tilde{A}_k^*(\cdot)$ by

$$(\tilde{A}_k^*(x))_{ij} = \int_Z A_k(x, u(z))(e_i + \nabla_z w_k^i(z; x)) \cdot e_j \, dz,$$

where $w_k^i(\cdot; x) \in H_{\#}^1(Z)/\mathbb{R}$ is the solution of

$$\nabla_z \cdot \left(A_k(x, u(z))(e_i + \nabla_z w_k^i(z; x)) \right) = 0.$$

This means that $\tilde{A}_k^*(x) \in \overline{G_x(A_k)}$. Thus, similar to the proof of (2.23), we have

$$\begin{aligned} & |(A_k^*(x))_{ij} - (A^*(x))_{ij}| \\ & \leq |(\tilde{A}_k^*(x))_{ij} - (A^*(x))_{ij}| \\ & \leq \int_Z |A(x, u(z))(\nabla_z w^i(z; x) - \nabla_z w_k^i(z; x))| \, dz \\ & \quad + \int_Z |(A_k(x, u(z)) - A(x, u(z)))(e_i + \nabla_z w_k^i(z; x))| \, dz \\ & \leq \left(\int_Z |A_k(x, u(z)) - A(x, u(z))|^q \, dz \right)^{1/q}. \end{aligned} \quad (2.32)$$

By Lebesgue's dominated convergence theorem, we deduce

$$\lim_{k \rightarrow \infty} |(A_k^*(x))_{ij} - (A^*(x))_{ij}| = 0, \quad \text{a.e. } x \in \Omega, \quad (2.33)$$

which proves (2.31).

Furthermore, noting that $A_k^*(\cdot)$ is piecewise constant and $A_k^*(\cdot) \in \mathcal{P}(A_k)$, by Step II, $A_k^*(\cdot) \in \mathcal{G}(A_k)$. Then there exists $u^{k,j}(\cdot) \in \mathcal{U}$, such that

$$A_k(\cdot, u^{k,j}(\cdot)) \xrightarrow{H} A_k^*(\cdot), \quad (j \rightarrow +\infty). \quad (2.34)$$

By Proposition 2.2, we can suppose that

$$A(\cdot, u^{k,j}(\cdot)) \xrightarrow{H} A_k(\cdot), \quad (j \rightarrow +\infty). \quad (2.35)$$

By Proposition 2.7, we obtain

$$\begin{aligned}
& \|A_k^*(\cdot) - A_k(\cdot)\|_{L^1(\Omega)} \\
& \leq C \lim_{j \rightarrow +\infty} \|A_k(\cdot, u^{k,j}(\cdot)) - A(\cdot, u^{k,j}(\cdot))\|_{L^1(\Omega)} \\
& = C \lim_{j \rightarrow +\infty} \int_{\Omega} \left| \sum_{\ell=1}^k \frac{1}{|\Omega_{\ell}^k|} \int_{\Omega_{\ell}^k} \left(A(s, u^{k,j}(x)) - A(x, u^{k,j}(x)) \right) ds \chi_{\ell}^k(x) \right| dx \\
& = C \lim_{j \rightarrow +\infty} \sum_{\ell=1}^k \frac{1}{|\Omega_{\ell}^k|} \int_{\Omega_{\ell}^k} \left| \int_{\Omega_{\ell}^k} \left(A(s, u^{k,j}(x)) - A(x, u^{k,j}(x)) \right) ds \right| dx \\
& \leq C \sum_{\ell=1}^k \frac{1}{|\Omega_{\ell}^k|} \int_{\Omega_{\ell}^k} \int_{\Omega_{\ell}^k} \omega(|F(s) - F(x)|) ds dx.
\end{aligned}$$

Thus it follows from Lemma 2.8 that,

$$\lim_{k \rightarrow +\infty} \|A_k^*(\cdot) - A_k(\cdot)\|_{L^1(\Omega)} = 0.$$

Combining the the above with (2.33), we get

$$\lim_{k \rightarrow +\infty} \|A_k(\cdot) - A^*(\cdot)\|_{L^1(\Omega)} = 0.$$

Consequently,

$$A_k(\cdot) \xrightarrow{H} A^*(\cdot). \quad (2.36)$$

It follows from $A_k(\cdot) \in \mathcal{G}(A)$ that $A^*(\cdot) \in \mathcal{G}(A)$. This ends the proof. \square

3 Proof of the Main Theorem

In this section, we will prove our main result. Before that, we need to show three lemmas. The first is about the well-posedness and regularity of state equation (1.1).

Lemma 3.1. *Let (S1)–(S4) hold. Then for any $u(\cdot) \in \mathcal{U}$, (1.1) admits a unique weak solution $y(\cdot) \in H_0^1(\Omega) \cap L^\infty(\Omega)$. Furthermore, there exists a constant $R > 0$, independent of $u(\cdot)$, such that*

$$\|y\|_{H_0^1(\Omega)} + \|y\|_{L^\infty(\Omega)} \leq R. \quad (3.1)$$

The existence of a weak solution to (1.1) in $H_0^1(\Omega)$ together with the $H_0^1(\Omega)$ -norm estimate follows easily from the variational structure of (1.1), while the uniqueness of the weak solution follows from (S3) and (1.5). The boundedness of weak solution in $L^\infty(\Omega)$ follows from standard De Giorgi iteration.

In order to proof our main theorem, we need another lemma.

Lemma 3.2. Assume $A^\varepsilon(\cdot) \in L^\infty(\Omega, \mathcal{M}(\Lambda, \lambda))$ and

$$A^\varepsilon(\cdot) \xrightarrow{H} A^*(\cdot), \quad (\varepsilon \rightarrow 0^+).$$

Moreover $f^\varepsilon(\cdot), f(\cdot) \in L^2(\Omega)$ and $f^\varepsilon(\cdot) \rightharpoonup f(\cdot)$ weakly in $L^2(\Omega)$. Let $y^\varepsilon(\cdot) \in H_0^1(\Omega)$ be the weak solution of

$$\begin{cases} -\nabla \cdot (A^\varepsilon(x) \nabla y^\varepsilon(x)) = f^\varepsilon(x), & \text{in } \Omega, \\ y^\varepsilon(x) = 0, & \text{on } \partial\Omega. \end{cases} \quad (3.2)$$

Then

$$y^\varepsilon(\cdot) \rightharpoonup \bar{y}(\cdot) \quad \text{weakly in } H_0^1(\Omega),$$

where $\bar{y}(\cdot)$ is the weak solution of

$$\begin{cases} -\nabla \cdot (A^*(x) \nabla \bar{y}(x)) = f(x), & \text{in } \Omega, \\ \bar{y}(x) = 0, & \text{on } \partial\Omega. \end{cases} \quad (3.3)$$

Proof. Set $h^\varepsilon(\cdot) = f^\varepsilon(\cdot) - f(\cdot)$. Then $\|h^\varepsilon(\cdot)\|_{L^2(\Omega)}$ is bounded. Let $z^\varepsilon(\cdot) \in H_0^1(\Omega)$ be the weak solution of

$$\begin{cases} -\nabla \cdot (A^\varepsilon(x) \nabla z^\varepsilon(x)) = h^\varepsilon(x), & \text{in } \Omega, \\ z^\varepsilon(x) = 0, & \text{on } \partial\Omega. \end{cases} \quad (3.4)$$

We have

$$\begin{aligned} \lambda \int_{\Omega} |\nabla z^\varepsilon(x)|^2 dx &\leq \int_{\Omega} A^\varepsilon(x) \nabla z^\varepsilon(x) \cdot \nabla z^\varepsilon(x) dx \\ &= \int_{\Omega} z^\varepsilon(x) h^\varepsilon(x) dx \leq C \|z^\varepsilon(\cdot)\|_{L^2(\Omega)}. \end{aligned}$$

Hence $\|z^\varepsilon(\cdot)\|_{H_0^1(\Omega)}$ is bounded. Then along a subsequence $\varepsilon \rightarrow 0^+$,

$$z^\varepsilon(\cdot) \rightarrow \bar{z}(\cdot), \quad \text{weakly in } H_0^1(\Omega), \text{ strongly in } L^2(\Omega).$$

Consequently, along a subsequence $\varepsilon \rightarrow 0^+$,

$$\lambda \int_{\Omega} |\nabla z^\varepsilon(x)|^2 dx \leq \int_{\Omega} z^\varepsilon(x) h^\varepsilon(x) dx \rightarrow 0,$$

which means

$$z^\varepsilon(\cdot) \rightarrow \bar{z}(\cdot) = 0, \quad \text{si } H_0^1(\Omega).$$

Moreover, we can get that $z^\varepsilon(\cdot)$ itself converges to 0 strongly in $H_0^1(\Omega)$.

Since

$$\begin{cases} -\nabla \cdot (A^\varepsilon(x) \nabla (y^\varepsilon(x) - z^\varepsilon(x))) = f(x), & \text{in } \Omega, \\ y^\varepsilon(x) - z^\varepsilon(x) = 0, & \text{on } \partial\Omega, \end{cases} \quad (3.5)$$

$$y^\varepsilon(\cdot) - z^\varepsilon(\cdot) \rightharpoonup \bar{y}(\cdot), \quad \text{weakly in } H_0^1(\Omega).$$

Thus

$$y^\varepsilon(\cdot) = (y^\varepsilon(\cdot) - z^\varepsilon(\cdot)) + z^\varepsilon(\cdot) \rightharpoonup \bar{y}(\cdot), \quad \text{weakly in } H_0^1(\Omega).$$

This ends the proof. \square

The third lemma is about relaxed control defined by finite-additive probability measures. Denote $C(U)$ the bounded continuous function space on U , and $\mathcal{M}(U)$ the space of all regular bounded finitely additive measures on U . Moreover, denote

$$\mathcal{M}_+^1(U) = \left\{ \mu \in \mathcal{M}(U) \mid \mu \text{ is nonnegative and } \mu(U) = 1 \right\}$$

and

$$\mathcal{R}(\Omega, U) = \left\{ \sigma : \Omega \rightarrow \mathcal{M}_+^1(U) \mid x \mapsto \int_U h(v) \sigma(x)(dv) \text{ is measurable, } \forall h \in C(U) \right\}.$$

Let $C(U)^*$ and $L^1(\Omega; C(U))^*$ be the dual spaces of $C(U)$ and $L^1(\Omega; C(U))$, respectively. We regard $\mathcal{M}_+^1(U)$ and $\mathcal{R}(\Omega, U)$ as subspace of $C(U)^*$ and $L^1(\Omega; C(U))^*$ by setting

$$\mu(h) \triangleq \int_U h(v) \mu(dv), \quad \forall h \in C(U), \quad (3.6)$$

and

$$\sigma(g) \triangleq \int_\Omega \int_U h(x, v) \sigma(x)(dv), \quad \forall g \in L^1(\Omega; C(U)). \quad (3.7)$$

By Theorems 12.2.11 and 12.4.6 in [9], (3.6) and (3.7) are well defined. Thus we denote $\sigma_k(\cdot) \xrightarrow{\mathcal{R}} \sigma(\cdot)$ if

$$\lim_{k \rightarrow \infty} \int_\Omega \int_U h(x, v) \sigma_k(x)(dv) dx = \int_\Omega \int_U h(x, v) \sigma(x)(dv) dx, \quad \forall h \in L^1(\Omega; C(U)). \quad (3.8)$$

We have (see Theorem 12.5.9 in [9]):

Lemma 3.3. *Assume (S1) — (S2) hold. Let $u_k(\cdot)$ be a sequence in \mathcal{U} . Then there is a subsequence of $u_k(\cdot)$, still denote by itself, such that*

$$\delta_{u_k(\cdot)} \xrightarrow{\mathcal{R}} \sigma(\cdot)$$

for some $\sigma(\cdot) \in \mathcal{R}(\Omega, U)$, i.e.

$$\lim_{k \rightarrow \infty} \int_\Omega h(x, u_k(x)) dx = \int_\Omega \int_U h(x, v) \sigma(x)(dv) dx, \quad \forall h \in L^1(\Omega; C(U)). \quad (3.9)$$

Now we are at the position to prove Theorem 1.1.

Proof of Theorem 1.1. Let $u_k(\cdot) \in \mathcal{U}$ be a minimizing sequence of Problem (C), $y_k(\cdot)$ be the corresponding state sequence. Then

$$\|y_k(\cdot)\|_{H_0^1(\Omega)} + \|y_k(\cdot)\|_{L^\infty(\Omega)} \leq R.$$

Thus, along a subsequence,

$$y_k(\cdot) \rightarrow \bar{y}(\cdot) \quad \text{weakly in } H_0^1(\Omega), \quad \text{a.e. in } \Omega \quad (3.10)$$

for some $\bar{y}(\cdot) \in H_0^1(\Omega) \cap L^\infty(\Omega)$.

By Proposition 2.2, there exists an $A^*(\cdot) \in L^\infty(\Omega; \mathcal{M}(\Lambda, \lambda))$ and a subsequence of $u_k(\cdot)$, still denoted by $u_k(\cdot)$, such that

$$A(\cdot, u_k(\cdot)) \xrightarrow{H} A^*(\cdot).$$

Then by (2.24), along a subsequence $h \rightarrow 0$,

$$A_{ij}^*(x) = \lim_{h \rightarrow 0} \lim_{k \rightarrow \infty} \int_Z A(x, u_k(x + hz))(e_i + w_{h,k}^i(z; x)) \cdot e_j \, dz. \quad (3.11)$$

where $w_{h,k}^i(\cdot; x) \in H_{\#}^1(Z)$ is the Z -periodic solution of

$$\nabla \cdot \left(A(x, u_k(x + hz))(e_i + w_{h,k}^i(z; x)) \right) = 0.$$

On the other hand, since

$$|f(x, y_k(x), u_k(x))| \leq M_R,$$

we can suppose that

$$f(\cdot, y_k(\cdot), u_k(\cdot)) \rightharpoonup \bar{f}(\cdot), \quad \text{weakly in } L^2(\Omega) \quad (3.12)$$

for some $\bar{f}(\cdot) \in L^\infty(\Omega)$.

In order to characterize \bar{f} precisely, it is useful to use relax controls defined by finite-additive measures. By Lemma 3.3, we can suppose that

$$\delta_{u_k(\cdot)} \xrightarrow{\mathcal{R}} \sigma(\cdot)$$

for some $\sigma(\cdot)$ in $\mathcal{R}(\Omega, U)$. That is,

$$\lim_{k \rightarrow \infty} \int_\Omega h(x, u_k(x)) \, dx = \int_\Omega \int_U h(x, v) \sigma(x)(dv) \, dx, \quad \forall h \in L^1(\Omega; C(U)). \quad (3.13)$$

In particular, for any $g \in L^2(\Omega)$,

$$\lim_{k \rightarrow \infty} \int_\Omega f(x, \bar{y}(x), u_k(x)) \, dx = \int_\Omega \int_U f(x, \bar{y}(x), v) \sigma(x)(dv) \, dx.$$

That is,

$$f(x, \bar{y}(x), u_k(x)) \rightharpoonup \int_U f(x, \bar{y}(x), v) \sigma(x)(dv), \quad \text{weakly in } L^2(\Omega).$$

On the other hand, by (S4) and (3.10),

$$\begin{aligned} & |f(x, y_k(x), u_k(x)) - f(x, \bar{y}(x), u_k(x))| \\ & \leq M_R |y_k(x) - \bar{y}(x)| \rightarrow 0, \quad (k \rightarrow \infty). \end{aligned}$$

Therefore,

$$f(x, y_k(x), u_k(x)) \rightharpoonup \int_U f(x, \bar{y}(x), v) \sigma(x)(dv), \quad \text{weakly in } L^2(\Omega),$$

i.e.

$$\bar{f}(x) = \int_U f(x, \bar{y}(x), v) \sigma(x)(dv). \quad (3.14)$$

Furthermore, define $u_k(x) = 0$ if $x \notin \Omega$, then for almost all $x \in \Omega$,

$$\begin{aligned} & \left| \int_Z f(x, y_k(x), u_k(x + hz)) dz - \int_Z f(x, \bar{y}(x), u_k(x + hz)) dz \right| \\ & \leq M_R |y_k(x) - \bar{y}(x)| \rightarrow 0, \quad (k \rightarrow \infty) \end{aligned} \quad (3.15)$$

and

$$\begin{aligned} & \lim_{k \rightarrow \infty} \int_Z f(x, \bar{y}(x), u_k(x + hz)) dz \\ & = \lim_{k \rightarrow \infty} \frac{1}{h^n} \int_{\Omega} f(x, \bar{y}(x), u_k(z)) \chi_{x+hZ}(z) dz \\ & = \frac{1}{h^n} \int_{\Omega} \int_U f(x, \bar{y}(x), v) \chi_{x+hZ}(z) \sigma(z)(dv) dz \\ & = \int_Z \int_U f(x, \bar{y}(x), v) \sigma(x + hz)(dv) dz. \end{aligned} \quad (3.16)$$

Combing (3.14), (3.15) and (3.16), we obtain

$$\lim_{h \rightarrow 0} \lim_{k \rightarrow \infty} \int_Z f(x, y_k(x), u_k(x + hz)) dz = \bar{f}(x), \quad \text{a.e. } x \in \Omega. \quad (3.17)$$

In addition, we define

$$\begin{aligned} f_{k,h}^0(x) &= \int_Z f^0(x, y_k(x), u_k(x + hz)) dz, \\ \bar{f}^0(x) &= \varliminf_{h \rightarrow 0} \varliminf_{k \rightarrow \infty} f_{k,h}^0(x), \quad \text{a.e. } x \in \Omega. \end{aligned} \quad (3.18)$$

Then, combining (3.11), (3.17) with (3.18), we obtain that along a subsequence $h \rightarrow 0$,

$$\begin{pmatrix} A_{ij}^*(x) \\ \bar{f}(x) \\ \bar{f}^0(x) \end{pmatrix} = \lim_{h \rightarrow 0} \lim_{k \rightarrow \infty} \begin{pmatrix} \int_Z A(x, u_k(x + hz))(e_i + w_{h,k}^i(z)) \cdot e_j dz \\ \int_Z f(x, y_k(x), u_k(x + hz)) dz \\ \int_Z f^0(x, y_k(x), u_k(x + hz)) dz \end{pmatrix}. \quad (3.19)$$

Thus,

$$(A^*(x), \bar{f}(x), \bar{f}^0(x)) \in \bigcap_{\delta > 0} \overline{GE(x, B_\delta(\bar{y}(x)))}, \quad \text{a.e. } x \in \Omega. \quad (3.20)$$

By (1.10),

$$(A^*(x), \bar{f}(x), \bar{f}^0(x)) \in \mathcal{E}(x, \bar{y}(x)), \quad \text{a.e. } x \in \Omega. \quad (3.21)$$

Define

$$g(x, u) = |A(x, u) - A^*(x)| + |f(x, \bar{y}(x), u) - \bar{f}(x)| + [f^0(x, \bar{y}(x), u) - \bar{f}^0(x)]^+,$$

where a^+ denote the positive part of a real number a . Then, $g(x, u)$ is measurable in x and continuous in u . It follows from (3.21) and the definition of $\mathcal{E}(x, \bar{y}(x))$ that $0 \in g(x, U)$. By Filippov's lemma, there exists a $\bar{u}(\cdot) \in \mathcal{U}$, such that

$$\begin{cases} A^*(x) = A(x, \bar{u}(x)), \\ \bar{f}(x) = f(x, \bar{y}(x), \bar{u}(x)), \\ \bar{f}^0(x) \geq f^0(x, \bar{y}(x), \bar{u}(x)), \end{cases} \quad \text{a.e. } x \in \Omega.$$

Consequently, $\bar{y}(\cdot)$ is the weak solution of

$$\begin{cases} -\nabla \cdot (A(x, \bar{u}(x)) \nabla \bar{y}(x)) = f(x, \bar{y}(x), \bar{u}(x)), & \text{in } \Omega, \\ \bar{y}(x) = 0, & \text{on } \partial\Omega. \end{cases}$$

Finally, by Fatou's lemma,

$$\begin{aligned} J(\bar{u}(\cdot)) &= \int_{\Omega} f^0(x, \bar{y}(x), \bar{u}(x)) dx \\ &\leq \int_{\Omega} \bar{f}^0(x) dx = \int_{\Omega} \lim_{h \rightarrow 0} \lim_{k \rightarrow 0} f_k^0(x) dx \\ &\leq \lim_{h \rightarrow 0} \lim_{k \rightarrow 0} \int_{\Omega} f_k^0(x) dx \leq \lim_{h \rightarrow 0} \lim_{k \rightarrow 0} \int_{\Omega} \int_Z f^0(x, y_k(x), u_k(x + hz)) dz dx \\ &= \lim_{k \rightarrow 0} J(u_k(\cdot)) = \inf_{u(\cdot) \in \mathcal{U}} J(u(\cdot)). \end{aligned}$$

This means that $\bar{u}(\cdot)$ is a solution of Problem (C), proving Theorem 1.1. \square

Proposition 3.4. *Let (S1) — (S5) hold. If $A(x, u) \equiv A(x)$, then (1.10) is equivalent to*

$$E(x, y) = \bigcap_{\delta > 0} \overline{\text{co}} E(x, B_\delta(y)), \quad (3.22)$$

where

$$E(x, y) = \left\{ (\zeta, \zeta^0) \in \mathbb{R} \times \mathbb{R} \mid \zeta = f(x, y, u), \zeta^0 \geq f^0(x, y, u), u \in U \right\}.$$

Proof. Denote

$$\tilde{E}(x, y) = \left\{ (\zeta, \zeta^0) \in \mathbb{R} \times \mathbb{R} \mid \zeta = \int_Z f(x, y, u(z)) dz, \zeta^0 \geq \int_Z f^0(x, y, u(z)) dz, u(\cdot) \in \mathcal{U}_Z \right\}.$$

When $A(x, v) \equiv A(x)$, (1.10) is equivalent to

$$E(x, y) = \bigcap_{\delta > 0} \overline{\tilde{E}(x, B_\delta(y))}. \quad (3.23)$$

To prove (3.22), we need only to show that

$$\overline{\text{co}} E(x, y) = \overline{\tilde{E}(x, y)}. \quad (3.24)$$

I. We first prove $\overline{\text{co}} E(x, y) \subseteq \overline{\tilde{E}(x, y)}$.

For any $(\zeta, \zeta^0) \in \text{co } E(x, y)$, there exist $\alpha_i, i = 1, 2, \dots, m$ such that $\sum_{i=1}^m \alpha_i = 1$ and

$$\zeta = \sum_{i=1}^m \alpha_i f(x, y, u_i), \quad \zeta^0 \geq \sum_{i=1}^m \alpha_i f^0(x, y, u_i).$$

Define $u_0(\cdot) \in \mathcal{U}_Z$ by

$$u_0(z) = \begin{cases} u_1, & z_1 \in [0, \alpha_1], \\ u_k, & z_1 \in \left(\sum_{i=1}^{k-1} \alpha_i, \sum_{i=1}^k \alpha_i \right]. \end{cases} \quad (3.25)$$

Thus

$$\int_Z f(x, y, u_0(z)) dz = \sum_{i=1}^m \alpha_i f(x, y, u_i) = \zeta$$

and

$$\int_Z f^0(x, y, u_0(z)) dz = \sum_{i=1}^m \alpha_i f^0(x, y, u_i) \leq \zeta^0.$$

This means $(\zeta, \zeta^0) \in \tilde{E}(x, y)$. Thus $\text{co } E(x, y) \subseteq \tilde{E}(x, y)$, and then $\overline{\text{co}} E(x, y) \subseteq \overline{\tilde{E}(x, y)}$.

II. Now, we turn to prove $\overline{\tilde{E}(x, y)} \subseteq \overline{\text{co}} E(x, y)$.

Let $\{U_i^k\}_{1 \leq i \leq k}$ be a family of measurable decompositions of U , such that

(a) if $i \neq j$, then $U_i^k \cap U_j^k = \emptyset$;

(b) for any k , $\bigcup_{j=1}^k U_j^k = U$;

(c) $\lim_{k \rightarrow +\infty} \max_{1 \leq j \leq k} \text{diam}(U_j^k) = 0$.

Moreover, let $u_i^k \in U_i^k$ and

$$Z_i^k = \left\{ z \in Z \mid u(z) \in U_i^k \right\}.$$

Then by the continuity of f and the lower semi-continuity of f^0 , for a.e. $(x, y) \in \Omega \times \mathbb{R}$,

$$\lim_{k \rightarrow \infty} \sum_{i=1}^k f(x, y, u_i^k) \chi_{Z_i^k}(z) = f(x, y, u(z)), \quad \text{a.e. } z \in Z$$

and

$$\lim_{k \rightarrow \infty} \sum_{i=1}^k f^0(x, y, u_i^k) \chi_{Z_i^k}(z) \geq f^0(x, y, u(z)), \quad \text{a.e. } z \in Z,$$

which means for a.e. $(x, y) \in \Omega \times \mathbb{R}$,

$$\lim_{k \rightarrow \infty} \sum_{i=1}^k f(x, y, u_i^k) m(Z_i^k) = \int_Z f(x, y, u(z)) dz,$$

and

$$\lim_{k \rightarrow \infty} \sum_{i=1}^k f^0(x, y, u_i^k) m(Z_i^k) \geq \int_Z f^0(x, y, u(z)) dz.$$

Noting that $m(E_i^k) \geq 0$ and $\sum_{i=1}^k m(Z_i^k) = 1$ for $k = 1, 2, \dots$, we deduce $\widetilde{E}(x, y) \subseteq \overline{\text{co}} E(x, y)$.

Consequently, $\overline{\widetilde{E}(x, y)} \subseteq \overline{\text{co}} E(x, y)$. This ends the proof. \square

Similar to the classical cases (see for example, Chapter 3, Proposition 4.3 in [11]), we have the following proposition

Proposition 3.5. *Assume that*

(S6) *For almost all $x \in \Omega$, $f(x, \cdot, v)$ is continuous uniformly in $v \in U$ and $f^0(x, \cdot, v)$ is lower semi-continuous uniformly in $v \in U$, i.e. for any $y \in \mathbb{R}$ and $\varepsilon > 0$, there exists a $\tau = \tau(x, y) > 0$, such that for any $\tilde{y} \in B_\tau(y)$,*

$$\begin{cases} |f(x, \tilde{y}, v) - f(x, y, v)| < \varepsilon, \\ f^0(x, \tilde{y}, v) > f^0(x, y, v) - \varepsilon, \end{cases} \quad \forall v \in U. \quad (3.26)$$

Then (1.10) is equivalent to

$$\mathcal{E}(x, y) = \overline{G\mathcal{E}(x, y)}, \quad \text{a.e. } (x, y) \in \Omega \times \mathbb{R}. \quad (3.27)$$

Proof. We will prove that

$$\overline{G\mathcal{E}(x, y)} = \bigcap_{\delta > 0} \overline{G\mathcal{E}(x, B_\delta(y))}. \quad (3.28)$$

In fact, we need only to show

$$\bigcap_{\delta > 0} \overline{G\mathcal{E}(x, B_\delta(y))} \subseteq \overline{G\mathcal{E}(x, y)}$$

since

$$\overline{G\mathcal{E}(x, y)} \subseteq \bigcap_{\delta > 0} \overline{G\mathcal{E}(x, B_\delta(y))}$$

holds obviously.

By (S6), for any $y \in \mathbb{R}^n$ and $\varepsilon > 0$, there exists a $\tau = \tau(x, y) > 0$, such that for any $\tilde{y} \in B_\tau(y)$, (3.26) holds. For any $\delta \in (0, \tau)$. Let $(P_\delta, \zeta_\delta, \zeta_\delta^0) \in G\mathcal{E}(x, B_\delta(y))$. That is

$$\begin{cases} (P_\delta)_{ij} &= \int_Z A(x, u^\delta(z))(e_i + \nabla w_\delta^i(z; x)) \cdot e_j \, dz, \\ \zeta_\delta &= \int_Z f(x, y^\delta, u^\delta(z)) \, dz, \\ \zeta_\delta^0 &\geq \int_Z f^0(x, y^\delta, u^\delta(z)) \, dz \end{cases} \quad (3.29)$$

for some $u^\delta(\cdot) \in \mathcal{U}_Z$ and $y^\delta \in B_\delta(y)$, where $w^i(\cdot; x) \in H_{\#}^1(Z)$ solves

$$\nabla_z \cdot \left(A(x, u^\delta(z))(e_i + \nabla w_\delta^i(z; x)) \right) = 0.$$

Thus by (3.26), we obtain

$$\begin{cases} |\zeta_\delta - \int_Z f(x, y, u^\delta(z)) \, dz| \leq \int_Z |f(x, y^\delta, u^\delta(z)) - f(x, y, u^\delta(z))| \, dz < \varepsilon, \\ \zeta_\delta^0 \geq \int_Z f^0(x, y^\delta, u^\delta(z)) \, dz > \int_Z f^0(x, y, u^\delta(z)) \, dz - \varepsilon. \end{cases} \quad (3.30)$$

That is, $(P_\delta, \zeta_\delta, \zeta_\delta^0) \in B_\varepsilon(G\mathcal{E}(x, y))$. Consequently,

$$G\mathcal{E}(x, B_\delta(y)) \subseteq B_\varepsilon(G\mathcal{E}(x, y)). \quad (3.31)$$

Therefore

$$\bigcap_{\delta > 0} \overline{G\mathcal{E}(x, B_\delta(y))} \subseteq \bigcap_{\varepsilon > 0} \overline{B_\varepsilon(G\mathcal{E}(x, y))} = \overline{G\mathcal{E}(x, y)}, \quad (3.32)$$

which ends the proof. \square

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